Notice on the Symmetrical Polyhedra of Geometry

Auguste Bravais

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To God.

Preface

Auguste Bravais was born on 23^{rd} August 1811 in Annonay, France, and died 30^{th} March 1863 in Le Chesnay. He studied at the Collège Stanislas, Paris and did his doctorate in 1837 in Lyon. He then joined the Navy, and then taught astronomy at the Faculté des Sciences at Lyon in 1841. In 1845 he became professor of physics at the École Polytechnique, Paris, and in 1854 admitted to the geography and navigation section of the Académie des Sciences in Paris.

Bravais derived in 1848 the 14 possible arrangements of points in space. His Études cristallographiques (1866) exhaustively analyses the geometry of molecular polyhedra.

The present work of his was originally published as *Mémoire sur les polyèdres de forme symétrique* in *Journal de mathématiques pures et appliquées* by *Liouvile*, pages 137–140 and 141–180, Volume 14, 1849. It appears again in 1866 in *Études cristallographiques* by A. Bravais, *im Verlage von Gauthier-Villars*, Paris, and then again in *Note sur les polyèdres symétrique de la géometrie*, Möbius, *Gesammelte Werke*, Volume 2, from page 363.

This translation was based on a German translation which appeared as *Abhandlungen über symmetrische polyeder*, Uebers. und in.–Leipzig, W. Engelmann (b. 1890), Oswalds Klassiker der exakten Wissenschaften. The copy I perused was the one kept at John Rylands Library of Manchester, at Deansgate. It is a part of the Partington Collection.

Though it is true that it would be better to translate this work from the original in French than from a translated version, this has not been possible for the practical fact that I have not yet found such a copy anywhere. On the other hand, that there should have existed a translation from French into German, even at that time and in Europe, where most people read both languages, shows the importance of this seminal work by Bravais.

The numbers enclosed within pairs of square brackets are, I think, the numbers of page the materials following them were found in the monograph as it originally appeared.

Kit Tyabandha, Ph.D.
Bangkok, May 2006

[13] Notice

on

the symmetrical polyhedra of geometry

by

A. Bravais

Professor at the polytechnic school

One calls two polyhedra symmetrical, if they are constructed in the same fashion, one above, the other below a plane, with the condition that their homologous vertices are equally farther apart from this plane and lie on the one plane normal to this plane (*Legendre*, Géometrie bk. 6).

I call inverse polyhedra two polyhedra, whose homologous vertices lie at the same distance from a given point, on the one straight line passing through this point, but on opposite side.

I will indicate the first one, as supposed given polyhedron with P and with p its inverse: One sees, that inverted P will be the inverse of p.

I will call symmetry pole of two polyhedra the point, through which all straight lines pass, which pairwise connect the homologous vertices of the two polyhedra.

If one imagine from the two polyhedra P, p a single polyhedron (P, p) built, then one calls the said point: symmetry centre of the polyhedron.

Statement I. *If the symmetry pole of the considered-fixed polyhedron P moves, then the inverse p changes into a new inverse p', and p is always led by a simple translation general in all vertices into p'.*

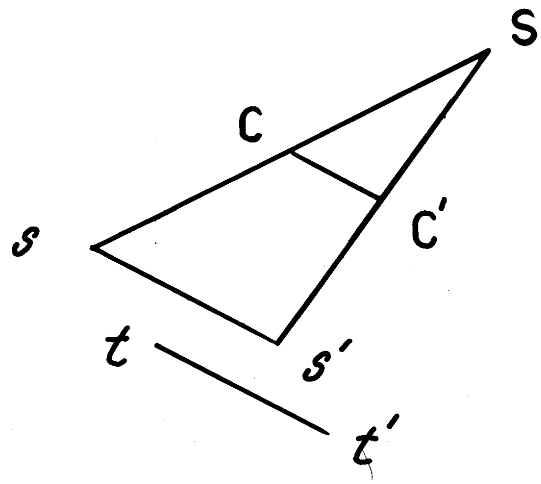

Figure 1

Let C, Fig. 1, be the first, C' the second symmetry pole, [14] and S an arbitrary vertex of the fixed polyhedron P. Let s be its homologous point of the shifting of C, and s' the same after this shifting.

From $Cs = CS$, $C's' = C'S$ [it] follows, that $ss' = 2CC'$, further, that ss' is parallel with CC'. Let likewise t and t' be the two locations, which another vertex of the inverse polyhedron takes after each other. One would have then in the same manner $tt' = 2CC'$, tt' parallel with CC'. If one then shifts the polyhedron, as one moves it in the direction from C towards C', precisely parallel with CC', and by an amount $= 2CC'$, then it reaches the agreement with the polyhedron p'. *q.e.d.*

Addendum. It follows from this, that, if a polyhedron P is given, its inverse, both the form of it, and the direction of its parts with regard to the absolute space concerned, is perfectly determined; however the location, that it will take, indeterminate and depends upon the location of the symmetry pole.

Statement II. *In two inverse polyhedra the homologous faces are pairwise equal to one another, and the inclination of two neighbouring faces in one of these two bodies is equal to the inclination of the homologous faces in the other.*

This Statement can be proved as Statement II of the sixth book by *Legendre*, where one takes vertex-triangle instead of the trapezoids with equal base used by the proof. But one can also proceed in simpler form as follows:

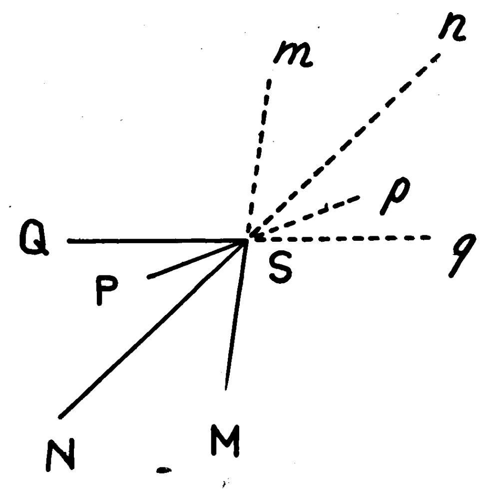

Figure 2

There be the prove, that both the edges, as well as the edge angles and the face angles, which makes the bodily vertex S, Fig. 2, are the same, as their homologous ones in the inverse polyhedron. Here the choice of the symmetry poles remains free, we take S as pole, extend the vertices SM, SN, SP, SQ by $Sm = SM$, $Sn = SN$, $Sp = SP$, $Sq = SQ$ etc. The two opposite bodily vertices will from construction apparently have edges pairwise the same among one another, their edge angles will be pairwise the same as vertex angles, and their face angles will also be as large as vertex angle; now both these edges, as well as the edge- and face-angles in the polyhedron and in its inverse are homologous to one another. The same proof can be employed for all bodily vertices, consequently also for all edge- and face-angles. Therefore the faces of two polyhedra are the same and equally inclined against one another. *q.e.d.*

[15] **Statement III.** *If one turns the polyhedron p, the inverse of P, by two right angles around a line passing through a symmetry pole C, then the polyhedron p', which one obtains in this manner, will be symmetrical to P with regard to the plane, which is placed through C normal to the rotation axis.*

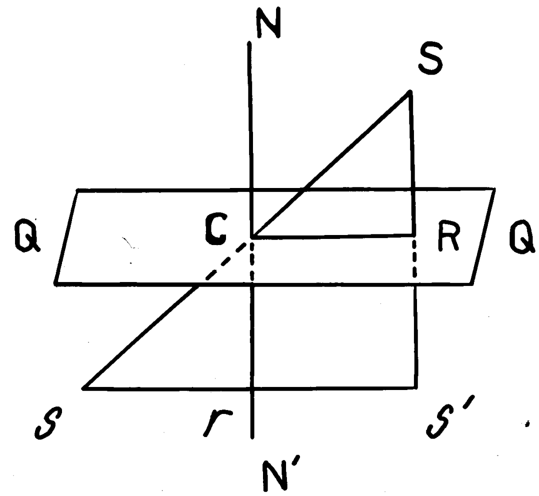

Figure 3

Let NN', Fig. 3, be the given straight line and QCQ' the plane placed perpendicular to it through C; let S be an arbitrary vertex of the polyhedron P, and s its homolog in the inverse polyhedron. One drops from s the perpendicular sr on NN' and extend the same to s' by the distance $rs' = rs$. It is obvious, that by a half turning around NN', s will come to s'. I join S with s'. Here $sC = CS$, $sr = rs'$, therefore Ss', will be parallel with NN' and consequently normal to the plane QQ'. If R is the intersection point of Ss' with the plane QQ', then the straight line CR will be normal to NN' and consequently parallel with ss'. Now one has $sC = CS$, therefore, also $SR = Rs'$. Consequently s' is the homologous vertex of S in the polyhedron, that is symmetrically constructed under the plane QQ' with regard to P. The same would be the case with every other vertex of the polyhedron P. Therefore a revolution of two right angles around NN' will bring the inverse p to agreement with p', the symmetrical of P with regard to the plane QQ'. *q.e.d.*

Remark. The plane QCQ' can be called the symmetry plane of the polyhedron (P, p'), if one takes (P, p') as a sole polyhedron.

Statement IV. *Inversely p' will also be the symmetrical of P with regard to an arbitrary plane QQ', if one turns it by two right angles around a normal to the plane, to become one of the inverses of P, and the symmetry pole lies then in the intersection point C of the plane and the rotation axis.*

Because if one constructs the inverse p, as one takes C to the symmetry pole, then p and p' can be brought into agreement, p with p' or p' with p, by a revolution of two right angles around NN'.

Addendum I. It follows from the two previous Statements, that the different polyhedra symmetrical to P, which their form falls upon, are nothing but its inverse polyhedron, and that the same, which the direction of their parts falls upon, are equivalent to the different polyhedra, which are obtained by revolution of the inverse p by two right angles around freely chosen axes.

[16] Two inverse polyhedra of a given polyhedron P are always identically directed in space. This does not hold with two symmetrical polyhedra of P, taken when the two symmetry planes, which they determine, are parallel to each other.

Addendum II. Two polyhedra, that are symmetrical to P with regard to two freely chosen planes, can always be brought into agreement, because they can both be brought into agreement with an inverse polyhedron P.

Statement V. *In two symmetrical polyhedra the homologous faces are pairwise the same with one another. (The remaining is as in Statement II.)*

This is the second Proposition of the 6^th book of *Legendre*. It is obvious, here the symmetrical polyhedron can always coincide with the inverse, and since the inverse, in relation to the original polyhedron, from our Statement II possesses the properties described here.

As for the rest the proof of this Statement is unnecessary.

Statement VI. *Two vertex corners are inverse to each other.*

In fact their common vertex is their symmetry pole.

Statement VII. *The plane, which goes through two opposite edges of a parallelepiped, divides it into two inverse prisms, and the bodily vertices lying opposite to one another are inverse to one another.*

One shows firstly, that the diagonals intersect one another in the same point. This point is therefore the symmetry centre of the parallelepiped.

If one then views the two prisms as two different polyhedra, then the symmetry centre will be a symmetry pole, therefore the bodily vertices lying opposite are also inverse, exactly the same as the two prisms. (Statements V and VI of the 6^th book of Géometrie by *Legendre*.)

Statement VIII. *On the sphere every spherical polygon P has its inverse p, the vertices of which lie diametrically opposite to those of the given polygon. If one turns p by 180° around a diameter of the sphere, then [17] the new polygon will be symmetrical to the polygon P with regard to the plane of that greatest circle, which is normal to the diameter chosen as rotary axis.*

This follows from this, that the middle point of the sphere is taken to the symmetry pole.

Inversely every symmetrical polygon of the spherical polygon P can be brought into agreement with its inverse p, via a half turning around the diameter normal to the symmetry plane.

Treatise on the polyhedra of symmetric form

by A. Bravais, Professor at the polytechnic school. (1849)

The number of the axes of the same kind will be given by the coefficients, which precede the symbolic letters for the axes; therefore the notation $\left[\Lambda^6, 3L^2, 3{L'}^2\right]$ will mean a six-time major axis, which is connected with three dual axes of a known kind, and three other dual axes of another kind.

The symmetry planes will be indicated by the letter Π, P, P'; we will take the letter Π for the symmetry plane, which is normal to the major axis Λ, the symbols P and P' for symmetry planes, which are perpendicular to no axis of the polyhedron, and the symbols $P^q {P'}^q P^{q'}$ for the symmetry planes, which lie normal to the axes L^q, ${L'}^q$, $L^{q'}$ of this polyhedron. The greatest number of kinds of these planes never exceed three.

The number of the symmetry planes of the same kind is, as with the axes, established by the coefficients which precedes the symbol of these planes. Therefore $\left[\Pi, 3P^{\frac{1}{2}}, 3{P'}^2\right]$ will then indicate: a symmetry plane, which is normal to the major axis, three symmetry planes of the same kind, normal to the axes $3L^2$, and three symmetry planes of another kind, which are normal to the axes $3{L'}^2$.

Definition IX and divisions. *With respect to their symmetry the polyhedra have been divided into four large classes:*

1. **Asymmetrical polyhedra**;
2. **Symmetrical polyhedra without axes**;

[26] 3. **Symmetrical polyhedra with major axis**; this class falls into two divisions: polyhedra with major axis of even order and polyhedra with major axis of odd order.

4. Symmetrical spheroidal polyhedra, which possess one or more axes, of which none is a major axis. These are divided into two groups; the quaternary polyhedra and the decemternary polyhedra, depending on the number of their singular ternary axes.

In the studies, which we will make on the polyhedra, we will totally disregard their faces and their edges, in order to look only at their vertices, so that each polyhedron for us will become a set of various points, the number of which is a limited one and which are spread around their centre of gravity in a certain manner.

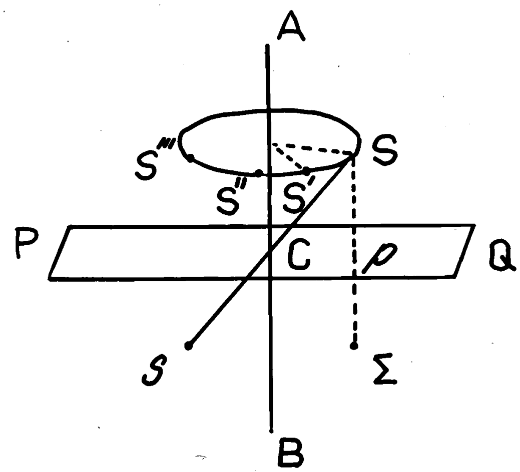

Figure 1

Definition I. I will call symmetry centre of a polyhedron a point C, Fig. 1, if, when I connect this point with any one vertex S of the polyhedron and extend CS by a magnitude equals to itself, the point s thus obtained is also a vertex of the polyhedron. This point s will be the homolog of S with regard to the middle point C.

Statement I. *In every bounded polyhedron there can arise only one symmetry centre.*

The truth of this assertion is evident.

Definition II. I will call symmetry axis of a polyhedron a straight line AB, Fig. 1, if with a revolution of the polyhedron by an angle Q around AB the new positions of the vertices coincide with the former ones. If for example this rotation brings the vertex S to S', then S' must also be the location of a vertex of the polyhedron, and then S and S' will be called homolog to each other with regard to the axis AB.

[22] **Statement II.** *The angle, which by a revolution of the polyhedron around a symmetry axis the locations of the vertices return to themselves, is always commensurate with* 360 *degrees.*

In deed: let S', Fig. 1, be a homolog of S with regard to the axis AB; one places through S and S' a plane normal to AB, which the axis will intersect with c; from c one describes with the radius cS a circle and makes

$$\operatorname{arc} S''S' = \operatorname{arc} SS', \quad \operatorname{arc} S'''S'' = \operatorname{arc} SS', \quad \textit{etc}$$

As the revolution takes the end point S from S to S', the vertex S', which takes part in this movement, will fall on S'', which will also be the location of a vertex. In this manner not only will S and S' be vertices of the polyhedron, but also S'' and another point S''', ..., which are found in the same way. As one repeats the arc SS' in the same fashion, one mus come back by one or more circuits to the start point S, otherwise the number of the vertices would be unbounded, which is not possible. Therefore, if one indicates the angle SCS' with K, and with p and q two relative primes, then:

$$K = \frac{p}{q} 360°.$$

Addendum. The smallest among the rotation angle, which is capable, to bring the locations of the vertices back to themselves, is $\frac{360°}{q}$. In fact the general expression for this angle certainly is

$$mK - n \cdot 360° = \frac{mp - nq}{q} 360°,$$

wherein m and n are integers. Now one can then always fix m and n, that they satisfy the condition

$$mp - nq = \pm 1,$$

consequently *etc.*

Definition III. One can therefore define the symmetry axis as one such straight line, around which a revolution, in an aliquot part $\frac{1}{q}$ of 360°, the position of the vertices of the polyhedron does not seem to appear different.

[23] The denominator q shall be called the order number of the symmetry of the axis. For $q = 2$, the axis shall be called symmetry axis of the second order or binary†

Statement III. *If two or more symmetry axes exist, then these axes and the symmetry planes, which the polyhedron sometimes has, all intersect one another in the same point.*

Because the centre of gravity of the vertices of the polyhedron, if we take these as equally heavy, must obviously, from the well known construction of the point of gravity, lie on every symmetry axis and also on all symmetry planes of the polyhedron.

Definition VI. The point, in which the axes and planes of symmetry of the polyhedron mutually meet, shall be called centre of the form‡ of the polyhedron. When one sole symmetry axis exists, if all symmetry planes pass through [24] this axis and no symmetry centre exists, then as few exists a centre of the form.

The centre of a regular tetrahedron is a centre of the form, but no symmetry centre for this polyhedron.

Definition VII. Two symmetry axes of the same order shall be called the same kind, if the arrangement of the vertices around the one the same [sic] is as around the other. In order to establish the conformity of this arrangement, one connects in imagination the vertices of the polyhedron with the two axes, and imagine the one of these two systems movable. If then at the same time the movable axis can be brought to the cover with the fixed one and the movable vertices with the fixed ones, then the axes are called of the same kind and directly different.

If the polyhedra, when the axes coincide, can not be brought to agreement, if they are only homologous with reference to a known symmetry plane, ie. if in the meaning taken from *Legendre* (Éléments de Géométrie) one is symmetrical to the other, then the axes will always still be of the same kind, but they shall then be called inversely different.

† The description binary [binär], ternary [ternär] *etc* are used in stead of the preferable translation of *Sohncke*'s (dual [zweizählig], triple [dreizählig] *etc*), in order to be able to accommodate the combinations as quaterternary [quaterternär] and decemternary [decemternär].

‡ Centre of the form (centre de figure) one has now coined the usual expression *geometrical middle point*, in order to preserve as much as possible the contrast with symmetry centre (centre de symétrie).

Two symmetry planes are of the same kind and directly different, if when one turns one of the two around their common intersection line and the polyhedron has taken part, the location of the vertices reach agreement, so long as the first-known symmetry plane coincides with the other. If otherwise the symmetrical conformity of the geometer takes the place of agreement conformity, then the symmetry planes will be called of the same kind, but inversely different.

If these two kinds of difference fail, then one calls the axes or planes of symmetry such ones of different kind.

Two axes of the same kind are necessarily of the same order; but the opposite is not necessarily the case.

Definition VIII. I call major axis a symmetry axis, to the whole other axes, if it gives still others, [which] are normal, and to the whole symmetry planes, if it gives those, [which] lie parallel or normal—suppose more over, that the order of symmetry of this major axis be no lower than that of the symmetry of the other axes.

If two or more axes satisfy these conditions, then one [25] of them can be arbitrarily chosen and looked upon as major axis of the polyhedron.

Notations. In order to symbolically establish the different kinds of symmetry, which the polyhedra can possess, I choose the letter C, to indicate a symmetry centre; $0C$ shall mean, that the polyhedron has no such centre.

The letters Λ, L, L' shall indicate symmetry axes; Λ^2, L^2, L'^2 dual axes; Λ^3, L^3 ... triple and so fort, whereby the index on top gives the order number of the symmetry.

The letter Λ shall always refer to the major axis.

These notations suffice for the axes, as no more than three different kinds of them can occur in a polyhedron.

§ I. Asymmetrical polyhedra

Here these polyhedra neither have axes, nor centre, nor planes of symmetry, they can from the preceding definitions be represented by the symbol

$$[0L, 0C, 0P].$$

§ II. Symmetrical polyhedra without axes

Statement IV. *In every polyhedron, that possesses a symmetry plane and a symmetry centre, the straight line, which is placed through this centre normal to the plane, is a symmetry axis of even order.*

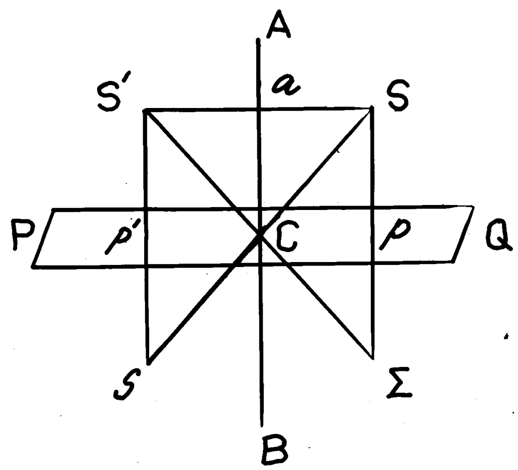

Figure 2

Let PQ, Fig. 2, be the symmetry plane, C the centre, S any vertex of the polyhedron, CaA the normal to this plane. Through this straight line and S one places the plane ACS normal to PQ, which contains the vertex s, the homolog of S with regard to the centre C, in exactly the same way as the vertex S', the homolog of s with regard to the plane PQ. If one S and S', then it is certain, that the connection straight-line will be perpendicular to CA and that $aS' = aS$ will hold. The condition for this, that the axis AC be a binary one, is thus fulfilled. The axis AC could again also be a quaternary, senary, in general, an axis of order $2q$. Consequently etc.

Statement V. *If two symmetry planes in a polyhedron exist, then their intersection line is a symmetry axis.*

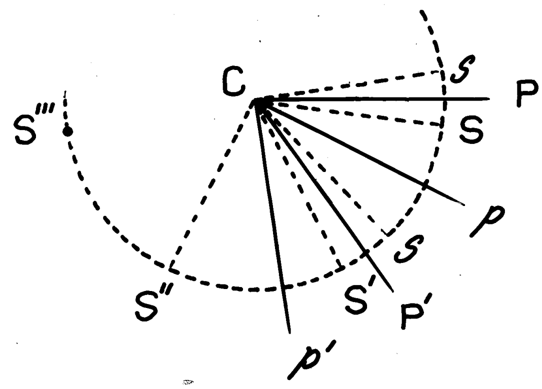

Figure 3

Let S, Fig. 3, be any vertex of the polyhedron. One takes the plane, which normal to the two given symmetry planes is placed through S, to the plane of the drawing; let CP and Cp be the trace of this plane on the plane of the drawing. One will obtain s, [27] the homolog of S with regard to the plane CP, as one makes in the plane PCp

$$\angle SCs = 2 \cdot \angle SCP = 2 \cdot \angle sCP, \qquad \text{and} \qquad Cs = CS.$$

In the same way one will obtain the homolog S' of s with regard to the plane Cp, if one makes

$$\angle SCs = 2 \cdot \angle sCp \qquad \text{and} \qquad CS' = Cs,$$

from which by subtraction follows:

$$S'CS = 2PCp, CS' = CS.$$

When one repeats the construction, which is done with S, with the vertex S', one will likewise come up with another homolog S'', which in exactly the same way is determined by the following equations for the pole:

$$S''CS' = 2PCp, CS'' = CS'.$$

If one then describes from C with a radius CS the circle $SS'S''$ and the arc SS' puts on a known number on this circumference, then the points $SS'S''$ etc, which necessarily make a bounded system, the vertices of a registered one, will be a polygon circumscribed by this circle. If q is the number of these vertices, then one sees, that to each point S, $q-1$ other points, that are homolog to S with regard to the normal to this plane, will correspond. This normal will therefore be a symmetry axis, whose order number q will depend on the value of the angle PCp.

Addendum. The angle SCS' is necessarily of the form

$$\frac{p}{q}360°,$$

wherein p and q are relative prime numbers, and in the case, where S and S' are two homologs as close to each other as possible, one has

$$SCS' = \frac{360°}{q},$$

from the addendum to Proposition II.

Statement VI. *The symmetrical polyhedra without axes have only two different kinds of symmetry, depending on whether they possess a symmetry centre or a symmetry plane.*

[28] Because they can not, from Statement IV, at the same time have a symmetry centre and a symmetry plane, even less so from Proposition V two symmetry planes. The symbols for these two kinds of symmetry will then be:

$$[0L, C, 0P]$$
$$[0L, 0C, P].$$

§ III. Symmetrical polyhedra with major axis

Statement VII. *If a polyhedron has two symmetry planes P and p, that are not normal to each other, then it has a third one P', which is the homolog of P with respect to the plane p and in the same kind as P.*

Let CP and Cp, Fig. 3, be the tracks of the two planes P and p on a plane normal to their mutual intersection line. Let CP' further be the track of the plane P' homologous to P, in relation with the plane Cp lying in between. The vertex s' will have a homolog in S with regard to Cp, and S a homolog s, on the other side of CP. Likewise s will have a homolog in S' of s' with respect to Cp; it is easy to see, that S' will be the homolog of s' with respect to the plane CP'. In this manner has each vertex one other homologous of its own, with regard to this last plane; therefore P' is also a symmetry plane and, as one sees, of the same kind as P.

Statement VIII. *If a polyhedron has a symmetry plane P and a symmetry axis L placed diagonal to it, then the straight line L', the homolog of L with respect to every plane, is also a symmetry axis.*

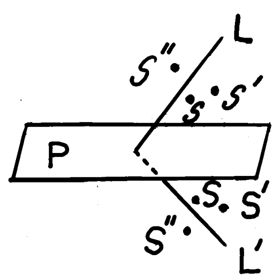

Figure 4

Because S, Fig. 4, usually is a vertex, which has in s a homolog with regard to the plane P, and s, s', s'', etc. present the system of the homologs of s with regard to the axis L. Let S, S', S'' etc. be the homologs of s, s', s'' etc. with respect to the plane P. The order of S, S', S'' etc. around L' will be the same as that of s, s', s'' etc. around L, therefore L' is also a symmetry axis and of the same kind as L.

Addendum. Every symmetry plane brings about, that not only every vertex, but also every [29] plane or axis of symmetry of the polyhedron repeats itself on the other side of the plane. The planes and axes, which come into being as such, are similarly of the same kind as the original ones and inverse to them.

Statement IX. *In every polyhedron, which has an axis of the order q, correspond to every symmetry plane placed diagonal to the axis $q-1$ other symmetry planes of the same kind.*

This Statement can be proved, as the Proposition VII and VIII.

One can also confine oneself to the notion, that during the revolution of the polyhedron around the axis L^q, the symmetry plane can be viewed as being involved in this revolution; now this plane, while the revolution takes place, does not cease, to be a symmetry plane for the polyhedron, which takes part in its motion. This goes therefore also for the case, that the revolution reaches a sum of $\frac{360°}{q}$.

Statement X. *In every polyhedron, which has an axis L of the order q, correspond to each symmetry axis L' placed diagonal to the first one, $q-1$ other axes of the same order and the same kind as the axis L'.*

This Statement could be proved as the previous one. If the polyhedron is turned by $\frac{360°}{q}$, then the axis does not cease, to be a symmetry axis of the moving polyhedron.

Addendum. Every symmetry axis of the order q presupposes the co-existence of all symmetry axes or planes, which are the homologs of a given axis or plane with regard to the axis of the order q. The homologous planes or axes obtained in this manner are of the same kind and directly different among themselves.

Statement XI. *If in total q symmetry planes exist which intersect one another in one and the same straight line, then this straight line is a symmetry axis, whose order number is q or a multiple of q.*

The face angles of these planes are necessarily equal among themselves, because without this these faces would symmetrically repeat with respect to one another (Statement VII), and their number would exceed q.

[30] Let therefore CPA and CpA, Fig. 5, be two neighbouring symmetry planes, then one will evidently have

$$PCp = \frac{180°}{q}.$$

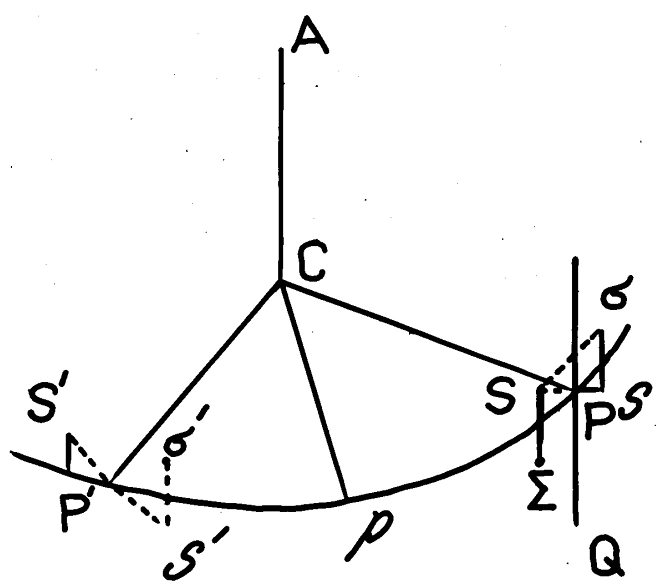

Figure 5

If now the point σ is the homolog of S with regard to the plane PCA, and S' the homolog of σ with regard to the plane pCA, then the revolution, which brings S to S', so that it revolves the polyhedron around CA, will be from Statement V. [sic]

$$PCP' = 2PCp = \frac{360°}{q}.$$

Therefore the axis CA will have the symmetry order q or a multiple of it.

Statement XII. *If the two binary symmetry axes exist, then the normal to their plane placed through their intersection point is a symmetry axis of the polyhedron.*

Let CP and Cp, Fig. 3, be the two binary axes, and S a vertex of the polyhedron. I will assume, that S lies at a height of Δ above the plane PCp, the plane of the drawing, and orthogonally projects itself in S; then the homolog of S with regard to the axis CP will be the point s, which lies at a distance Δ under the plane of the drawing, and one will have between the projection S and s the relationship

$$\angle SCs = 2\angle SCP \qquad \text{and} \qquad Cs = CS.$$

In the same way one will obtain the homolog S' of s with regard to the axis Cp, where one sets

$$\angle S'Cs = 2\angle sCP \qquad \text{and} \qquad CS' = Cs,$$

and the point S' will be at a height Δ above the plane of the drawing; therefore

$$S'CS = 2PCp \qquad \text{and} \qquad CS' = CS.$$

When one repeats the same operations with S', one obtains S'', then S'''. All these points will make the vertices of a regular polygon, which is registered to the circle of radius CS, and they will be the homologs of S with regard to the normal [31] to the plane. Therefore this normal will be a symmetry axis, whose order number q depends on the value of the angle PCp.

Statement XIII. *If on a plane is divided into a total number q of binary axes, then the normal to this plane is a symmetry axis, whose order number is q or a multiple of q.*

The angles of these axes are necessarily equal among themselves, otherwise they would oppositely reproduce themselves in the same plane (Statement X, Addendum) and their number would be greater than the number q.

Let therefore CP and Cp be two neighbouring binary axes, Fig. 5, and CA the normal to their plane, then one will evidently have:

$$PCp = \frac{180^\circ}{q}.$$

If now s is the homolog of S with regard to the axis CP, and S' the homolog of s with regard to the axis Cp, then the rotation, which brings S to S', where it turns the polyhedron around CA, from the explanation of the last Statement, will come to:

$$\angle PCP' = 2 \cdot PCp = \frac{360^\circ}{q}.$$

Therefore the straight line CA will be a symmetry axis of the order q or a multiple of q.

Statement XIV. *I three quaternary axes perpendicular to one another exist, then there exist at the same time four ternary axes outside of the planes, which pairwise bind the quaternary axes.*

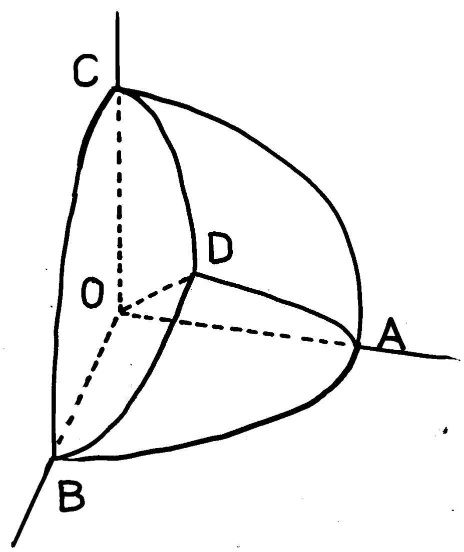

Figure 6

Let OA, OB and OC, Fig. 6, be the three quaternary axes, which meet the centre of the form of the polyhedron in O. We let the polyhedron make a quater-turn around OA admittedly from B to C. The apparent position of the vertex will remain the same, and the axis OB will come to OC. We let the polyhedron make a second quarter of a turn, that is around the vertical OC, from A to B. The apparent position of the vertex will still always be the same, the axis OB will remain in OC, and the axis OA will move to OB. The resultant of these double turn will therefore be, that the system of the [32] axes OA and OB fast bound to the polyhedron and moving with it will be moved on to the system of the fixed straight lines OB and OC. Now it is certain, that this changing of place amounts to a single turn of 120° around the straight line OD, which binds the centre of the form O with the middle point D, of the spherical triangle ABC having three right angles. One sees from this, that the line OD is a ternary axis of the polyhedron, and here there are eight triangles with three right angles, whose middle point pairwise lie opposite one another, thus give four such ternary axes, all outside of the planes AOB, AOC and BOC.

Addendum. A polyhedron with three quaternary axes perpendicular to one another has no major axis, because in every polyhedron, which has a major axis, at least two of the three angles, which build three arbitrary axis L, L' and L'', not lying in the same plane, from the definition of the major axis must be equal to 90°. Now the system of the three axes OA, OB and OD does not satisfy this condition.

Statement XV. *In every polyhedron, which has a major axis Λ^q, if a second symmetry axis exists, this axis, which necessarily lies in a plane normal to Λ^q, can only be of binary symmetry.*

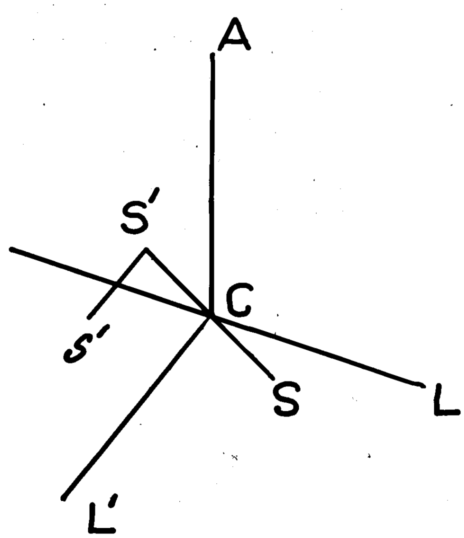

Figure 7

Let $C\Lambda$, Fig. 7, be the axis Λ^q, CL the second axis with the unknown order number of its symmetry x. It is clear, that this axis will lie in a plane normal to Λ^q, and I maintain, that $x = 2$ must hold.

In deed from the definition of the major axis the axes homologous to Λ^q with respect to L^x must (Statement X, Addendum) stand perpendicular to Λ^q, or coincide with it. Therefore one can only assume $x = 2$ or $x = 4$, which in the first case results from a half-, in the second case a quarter-turn around L^x.

If $x = 4$, then the axis CL will be quaternary, and the axis $C\Lambda$ will repeat itself in CL', which must be all axis of the same kind as itself (Statement X) and lies in the plane normal to $C\Lambda$ and passing through C. CL' will then also be an axis of the order q, and as a consequence the axis homologous to CL with respect to CL' does not go outside the plane LCL', as the definition of the major axis demands, therefore necessarily $q = 4$ or $q = 2$.

The case $q = 4$ corresponds with the case of the three perpendicular quaternary axis [33] and must be discarded, because it would give no major axis (Statement XIV, Addendum).

In the case $q = 2$ the axis CL will be the genuine major axis. It is in fact easy to see, that it can give no symmetry plane, which goes through $C\Lambda$ and lies oblique to $C\Lambda$, because its first homolog, with regard to the quaternary axis CL, would lie neither normal nor parallel to $C\Lambda$, and that would be inconsistent with the supposition, that $C\Lambda$ is a major axis. Nothing thus stand oppose to the supposition, that we look upon CL as major axis, and this we must do, since its symmetry order is a higher one, than the symmetry order of $C\Lambda$. Consequently, if $C\Lambda$ is really a major axis, it can never be that $x = 4$.

There remains therefore only the supposition $x = 2$; the second symmetry axis will consequently be a simple binary axis.

Statement XVI. *In every polyhedron with major axis the q binary axes, which lie in the plane normal to the major axis, are each equally inclined against the neighbouring ones and are by turns of the same kind.*

Because if CP an Cp, Fig. 3, be two neighbouring binary axes, then the binary axis Cp will force the axis CP, to repeat itself on the opposite side in CP' (Statement X, Addendum). CP' draws for itself the existence of the axis Cp' to itself, and so forth. From this one sees, that the q angles PCp, pCP', $P'Cp'$ etc. must be the same with one another and $= \frac{180^\circ}{q}$.

This axes CP and CP' are of the same kind and directly different from each other with respect to the axis Cp lying between them. Likewise Cp and Cp' are under the same kind of each other and each other directly different with respect to the axis CP', as a consequence etc.

Statement XVII. *The polyhedra, which have an axis Λ^q, further q symmetry planes, which pass through this axis, and the symmetry plane Π normal to the major axis, have at the same time q binary axes in the intersection lines of these q planes with the plane Π.*

The intersection of two symmetry planes is always a symmetry axis (Statement V); now this axis can only be binary (Statement XV), as a consequence etc.

One can, to prove this proposition, also recall Figure 5, where CPQ is one of the q planes, which pass through the axis Λ^q and CPP' the plane Π.

[34] Let Σ be the homolog of the vertex S with regard to the plane Π, let s be the homolog of Σ with regard to the plane CPQ. It is obvious, that S and s are homolog to each other with respect to CP, which therefore is a binary axis of the polyhedron.

Statement XVIII. *The polyhedron, which have an axis Λ^q, further q binary axes normal to Λ^q, and the symmetry plane Π normal to Λ^q, have also q symmetry planes, which pass through the axis Λ^q and through each of the binary axes.*

Let CP, Fig. 5, be the binary axis, let CPP' be the plane Π normal to the major axis CA. The vertex S will have a homolog in s on the other side of the binary axis CP; the vertex s will have a homolog in σ on the other side of the plane Π: the point S and σ will be homologous to each other with respact to the plane CPQ; consequently will finally be one of the symmetry planes of the polyhedron.

Statement XIX. *The polyhedron, which have a major axis Λ^q, further q symmetry axes, which pass through this axis, and a symmetry centre, have at the same time q binary axes, which lie through the centre normal to one of these planes.*

This is a consequence of the Statements IV and XV.

Statement XX. *The polyhedra, which have an axis Λ^q, further q binary axes, normal to Λ^q, and a symmetry centre, have also q symmetry planes, which are normal to these binary axes.*

This follows from Statement XXI, whose proof follows.

Polyhedra with major axis of even order

Statement XXI. *Every polyhedron, which has an axis of even order and a symmetry centre, has a symmetry plane, which passes through the centre and lies normal to each axis.*

Let CA, Fig. 2, be the symmetry axis of the order $2q$, C the symmetry centre and PQ the plane normal to CA. The vertex S has $2q-1$ homologs with respect to the axis CA; if one thus drops the normal Sa on to CA and extends it by $aS' = aS$, then S' will be one of these homologs. As one connects S' with the centre C and extend by $c\sigma = cS'$, one obtains the vertex Σ, the homolog of S' with respect to the middle point C. Apparently S and Σ are homologs with regard to the plane PQ; therefore this plane is a symmetry plane of the polyhedron.

Statement XXII. *Every polyhedron, which has an axis of even order and a symmetry plane, which lies normal to this axis, has a symmetry centre, which lies in the intersection point of the axis with the plane.*

Let S', Fig. 2, be the opposite homolog to the vertex S with regard to the axis CA. Let s be the homolog of S' with regard to the plane PQ normal to the axis CA. The two vertices S and s will be homologous with respect to the point C, in which axis and plane meet each other; this point must therefore be a symmetry centre of the polyhedron.

Statement XXIII. *Every polyhedron, which possesses an axis of even order and a symmetry plane, which goes through this axis, has at the same time a second symmetry plane, which passes through this axis and normal to the former plane.*

Let $C\Lambda$, Fig. 7, be the axis of even order, $LC\Lambda$ the given symmetry plane, S a vertex of the polyhedron, S' its homolog on the other side of $C\Lambda$ and s' the homolog of S' with respect to the plane $LC\Lambda$. S and s' will be homologous with respect to the plane $\Lambda CL'$, which passes through $C\Lambda$ and lies normal to the previous one; consequently etc.

Statement XXIV. *In every polyhedron, which possesses an axis L^{2q}, must, if binary, exist axes normal to L^{2q}, whose total number is equal to $2q$, namely q axes of a first kind and q axes of a second kind, which alternate with the first one.*

Let CP, Fig. 5, be a binary axis, which stands normal to the axis CA of the order $2q$; there will be in the plane PCP' normal to CA the straight line CP', which with CP makes the angle:

$$PCP' = \frac{360^\circ}{2q} = \frac{180^\circ}{q}.$$

The straight line CP' will be a binary axis of the same kind as CP, on account of the symmetry coming from the axis CA (Statement X, Addendum). The number of the axes thus obtained, [36] will be equal to q; each one of them can be taken as derivable from CP through the revolution

$$0^\circ, \frac{180^\circ}{q}, 2\frac{180^\circ}{q}, \ldots, (q-1)\frac{180^\circ}{q};$$

the axes, which contain the revolutions

$$q\frac{180^\circ}{q}, (q+1)\frac{180^\circ}{q}, \ldots, (2q-1)\frac{180^\circ}{q},$$

coincide with the former ones.

If one now divides PCP' by the bisecting Cp into two equal parts, then Cp will also be a binary axis. Because if s be the homolog of S with respect to CP, s' be the homolog of s with respect to the axis CA, then the revolution angle, which moves s into s' will be

$$\frac{360^\circ}{2q} = PCP',$$

the vertices S and s' be homologous with respect to Cp, because this will be seen as a binary axis. These q bisecting straight lines will therefore be binary axes, but of another type.

There exist from this $2q$ binary axes, and their number can not be greater; because if there were Q binary axes, and Q were greater than $2q$, then the symmetry order of the axis normal to their plane would be Q or mQ (Statement XIII), which goes against the above conditions.

Addendum I. The number of the binary axes, which are normal to a major axis Λ^{2q}, will always be $= 0$ or $2q$.

Addendum II. The binary axes normal to Λ^{2q} stand pairwise perpendicular to each other.

Statement XXV. *If in a polyhedron, which possesses an axis L^{2q}, symmetry planes passing through this axis exist, then their total number will be $2q$ and in deed q of one kind and, alternatively, q of another kind.*

[37] Let ACP, ACP', Fig. 5, be two symmetry planes, which make with each other the angles

$$PCP' = \frac{360^\circ}{2q} = \frac{180^\circ}{6}.$$

The system of this plane will be composed of q symmetry planes, which are of the same kind and directly different among one another. Moreover the planes lying among these, which divide the face angle (ACP, ACP') into two equal parts, are also symmetry planes. Because if ACp be one of the planes lying among these, if σ be the homolog of a given vertex S with respect to the plane ACP and σ' the homolog of σ with respect to CA. Here the revolution angle, which brings σ towards σ', takes on the values

$$\frac{360^\circ}{2q} = PCP',$$

therefore it is clear, that S and σ' lie symmetrical with respect to the plane ACp; therefore this plane is a symmetry plane, and there are in total q of which are of a type other than that of ACP, which are all directly different from ACp.

There are therefore $2q$ symmetry planes, which pass through L^{2q}, and their number can not be greater; because if it were equal to $Q > 2q$, then the symmetry order of the axis lying in their common intersection would be equal to Q or mQ (Statement XI), which is not possible.

Addendum I. The number of the symmetry planes, which pass through a major axis Λ^{2q}, is always 0 or $2q$, and can not be greater than this last number.

Addendum II. These symmetry planes stand pairwise perpendicular to one another.

Statement XXVI. *The polyhedron with major axis Λ^{2q}, which neither possess symmetry planes passing through this axis, nor binary axes, have two different kinds of symmetry, depending on whether the plane normal to the major axis is a symmetry plane or not.*

The number of the symmetry axes is determined by the contents of the statement. In exactly the same way [is] that of the symmetry planes, as soon as one knows, whether the [38] plane normal to the major axis is a symmetry plane or not. Whatever be the case with the existence of a symmetry centre, also hold true the same regarding a symmetry plane normal to the major axis hold (Statement XXI and XXII).

The symbol of these two kinds of symmetry will therefore be:

$$\left[\Lambda^{2q}, 0L^2, 0C, 0P\right]$$

$$\text{and} \quad \left[\Lambda^{2q}, 0L^2, C, \Pi\right].$$

Statement XXVII. *The polyhedra with major axis Λ^{2q}, which either have the $2q$ symmetry planes or only the $2q$ binary axes, which are connected with this major axis, can neither have a symmetry plane normal to the major axis, nor a symmetry centre.*

This is an evident consequence of Statements XVII, XVIII, XIX and XX.

The group of the polyhedra, on which the contents of the present Statement visit, falls into two classes of different symmetry, depending on whether the symmetry plane or the binary axes are absent. Here the binary axes of the two different kinds are (Statement XXIV), therefore we wish to indicate those of the first kind with L^2 and those of the second with L'^2. In the same fashion the symmetry planes shall be indicated with P, those of the other type with P'. One has from this, to represent the two classes of the polyhedron, the symbol:

$$\left[\Lambda^{2q}, qL^2, qL'^2, 0C, 0P\right]$$
$$\text{and} \quad \left[\Lambda^{2q}, 0L^2, 0C, qP, qP'\right]$$

Statement XXVIII. *In the polyhedra, which have the major axis $\Lambda^2 q$, further $2q$ binary axes and $2q$ symmetry planes passing through the major axis, the binary axes can be placed in symmetry planes, or alternate with them, ie. coincide with the bisection lines of their face angles. This gives for this particular case two different kinds of symmetry.*

For any other position of the axes and planes one against another the binary axes, as they bring forth anew the $2q$ symmetry planes through revolution of 180° (Statement X, Addendum), [39] would double the number of these last ones, and in the same way the number of the binary axes would double, since they would repeat themselves on the other side of the planes in homologous positions (Statement VIII, Addendum), which contradicts the addenda of XXIV and XXV.

Statement XXIX. *In the polyhedra with major axis Λ^{2q} there is, if the symmetry planes contain the binary axes, one symmetry plane normal to the major axis and one symmetry centre.*

Let CA, Fig. 5, be the major axis, CP one of the binary axes and $ACPQ$ the symmetry plane, which contains this axis. The vertex S will have its homolog with regard to the binary axis CP in s; s will have its homolog with regard to the plane $ACPQ$ in Σ. The relative position of S and Σ points to each other, that the plane PCP', normal to CA and to the plane CAP, is a symmetry plane of the polyhedron, with regard to which C and Σ are homologous vertices; therefore a symmetry plane normal to Λ^{2q} will exist, which by itself brings the existence of a symmetry centre with it (Statement XXII).

Statement XXX. *If in the polyhedra with major axis Λ^{2q} the symmetry planes alternate with the binary axes, then the latter are all of the same kind, but each inversely different from the neighbouring ones; then either a symmetry normal to the axis exists, or else a symmetry centre.*

If asymmetry plane normal to the axis exists, then its intersection lines with the symmetry planes would be binary axes (Statement XVII), which contradicts the contents of our Statement; therefore either a symmetry plane normal to the major axis exists, or a symmetry centre does (Statement XXI).

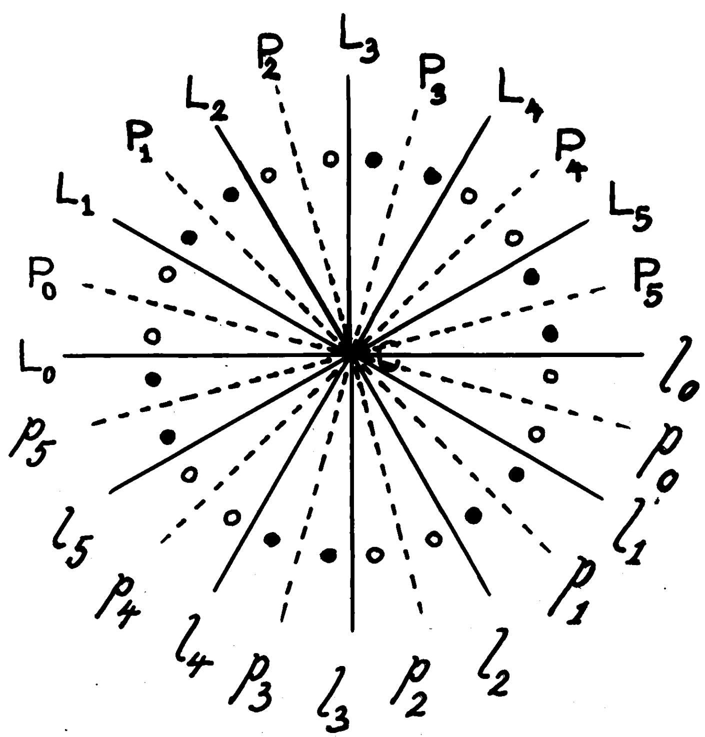

Figure 8

The neighbouring binary axes $L_0 l_0$ and $L_1 l_1$, Fig. 8, are inversely different from each other, homolog with regard to the symmetry plane lying in between, which goes through the major axis and through P_0Cp_0.

The inverse difference in general does not necessarily imply the direct difference; but in the present case it is easy, to be convinced, that L_0l_0 and L_1l_1 can never be directly different, because the revolution, which would lead to the characteristic agreement for direct difference, can realise in nature only either around the major axis or around the angle bisector p_0CP_0 [40] or lastly around the second angle bisector p_3CP_3 normal to the former one. Here $2q$ binary axes exist, therefore one has:

$$L_0CL_1 = \frac{180°}{2q} = \frac{360°}{4q} \qquad \text{and} \qquad P_0CP_1 = \frac{180°}{2q} = \frac{90°}{q}.$$

If CL_0 and CL_1 were directly different with respect to the major axis, then in this case the revolution, which reproduces the position of the vertices, would take on the value $\frac{360°}{4q}$, the symmetry order of the major axis $4q$, which is impossible. In the same way CL_0, CL_1 can not be directly different with regard to CP_0, since CP_0 is no binary axis. The normal to CP_0 ie. CP_3 is likewise at least a binary axis, since

$$90° = q \times P_0CP_1$$

holds, and since then this normal is a straight line of the groups CP_0, CP_1, CP_2. Therefore the neighbouring binary axes are inversely different, without any possibility exists, to bring them to agreement.

Fig. 8 shows the division of the homologous vertices for the case $2q = 6$. The symmetry plane, which lies normal to the major axis, is chosen as plane of the drawing. The filled-up small circles indicate the vertices, which lie under this plane, the ones left white those vertices, which lie above the same plane.

With the convention of the symbol for the two kinds of the symmetry sought after now is to notice, that in the case of Statement XXIX the symmetry planes, stand normal to the binary axes and that in the case, where the faces alternate with the axes, they can not be normal to them. We will therefore, 1. for the case of the coincidence, 2. for the case of the alternates referentially have the formulae:

$$\left[\Lambda^{2q}, qL^2, qL'^2, C, \Pi, qP^2, qP'^2\right]$$

$$\text{and} \quad \left[\Lambda^{2q}, 2qL^2, 0C, 2qP\right].$$

One sees from Statements XXVI, XXVII and XXVIII, that the polyhedra with major axis of even order can have only six kinds of symmetry. One will find them enumerated in the synoptic table at the end of this article.

[41] In this table one can give q all possible values, from $q = 1$ inclusively to $q = \infty$.

Polyhedra with major axis of odd order

Statement XXXI. *A polyhedron, which possesses a major axis of odd number, can not possess at the same time a symmetry plane normal to this axis and a symmetry centre.*

This is a consequence of Statement IV.

Statement XXXII. *If in a polyhedron, which possesses a major axis Λ^{2q+1}, there exist yet other axes, then these are binary, their total number is equal to $2q+1$, and they are all of the same kind.*

Any one of the new axes, which necessarily is binary, and lies in the plane normal to Λ^{2q+1} (Statement XV), repeats itself $2q$ times, conformably with the Addendum to Statement X; moreover these $2q+1$ axes will be different among themselves and not pairwise coincide (as that which happens in the case where the major axes are of even order). In deed, if one enumerate them with 0, 1, 2, etc. in sequence, as revolution following one after another, always let them take the values of $\frac{360°}{2q+1}$, then one finds, that the inclinations of 0, 1, 2, 3 etc. towards the axis O are

$$0°, \frac{360°}{2q+1}, \frac{2 \cdot 360°}{2q+1}, \ldots, \frac{q \cdot 360°}{2q+1}, \frac{(q+1) \cdot 360°}{2q+1}, \ldots, \frac{2q \cdot 360°}{2q+1},$$

and correspond to different straight lines, since two of these angles can never differ by 180°.

There can arise in the plane normal to the major axis no other binary axes, than those, whose existence we have just proved. Because were Q be their total number and were $Q > 2q+1$, then the order number of the symmetry of the major axis would be Q or mQ (Statement XIII), which contradicts the supposition of a major axis of the order $2q+1$.

Addendum. The number of the binary axes normal to Λ^{2q+1} must always be 0 or $2q+1$.

[42] **Statement XXXIII.** *In any polyhedron, which possesses the major axis Λ^{2q+1}, if at the same time symmetry planes passing through the major axis exist, these must all be of the same kind and their total number $2q+1$.*

One can point out as in the previous Statement: 1. that the $2q+1$ planes, which are each formed by the amount $\frac{360°}{2q+1}$ around the major axis, are of the same kind and directly different among themselves and not pairwise coincide; 2. that their number can not exceed $2q+1$ on the account of Statement XI.

Addendum. The number of the symmetry planes, which pass through the major axis Λ^{2q+1}, must always be equal to 0 or $2q+1$.

Statement XXXIV. *The polyhedra with major axis Λ^{2q+1}, which either possess symmetry planes passing through this axis, or binary axes, present three different kinds of symmetry, depending on whether they have a symmetry plane, which lies normal to this axis or not, and in this last case, depending on whether they have a symmetry centre or not.*

If the polyhedron possesses a symmetry plane normal to the major axis, then it can have no symmetry centre (Statement XXXI); then, here the axis is a major axis, the symmetry of the polyhedron is fully determined.

If the plane normal to the major axis is no symmetry plane, then the same impossibility no longer precedes. In this manner we will, according to our earlier sign, have the three different symbol:

$$\left[\Lambda^{2q+1}, 0L^2, 0C, 0P\right],$$

$$\left[\Lambda^{2q+1}, 0L^2, C, 0P\right],$$

$$\left[\Lambda^{2q+1}, 0L^2, 0C, \Pi\right].$$

Statement XXXV. *Those polyhedra with major axis Λ^{2q+1}, which either possess only the $2q+1$ symmetry planes or only the $2q+1$ binary axes, which are associable with every major axis, can neither have a symmetry plane lying normal to Λ^{2q+1} nor a symmetry centre.*

[43] This is an obvious consequence from the propositions XVII, XVIII, XIX and XX.

If one sticks to the signs already applied, then one finds, that the two classes of polyhedra, on which the present Statement relate, are represented by the following symbols:

$$\left[\Lambda^{2q+1}, (2q+1)L^2, 0C, 0P\right]$$

$$\text{and} \quad \left[\Lambda^{2q+1}, 0L^2, 0C, (2q+1)P\right].$$

Statement XXXVI. *In the polyhedra, which possess the major axis Λ^{2q+1}, further $2q+1$ binary axes and $2q+1$ symmetry planes going through the major axis, the binary axes lie on the symmetry planes or bisect their face angles.*

This Statement can be proved exactly as Statement XXVIII.

Statement XXXVII. *If in the polyhedra with major axis Λ^{2q+1}, the symmetry planes contain the binary axes, then a symmetry plane is normal to the major axis, but no centres of symmetry exist.*

The proof is the same as for Statement XXIX. But the conclusion related to the existence of the symmetry centre no longer hold on account of Statement XXXI.

Statement XXXVIII. *If in the polyhedra with major axis Λ^{2q+1} the symmetry planes alternate with the binary axes, then the latter are all of the same kind and coincide with the normals of the symmetry planes; there exists then a symmetry centre, but no symmetry plane normal to the major axis.*

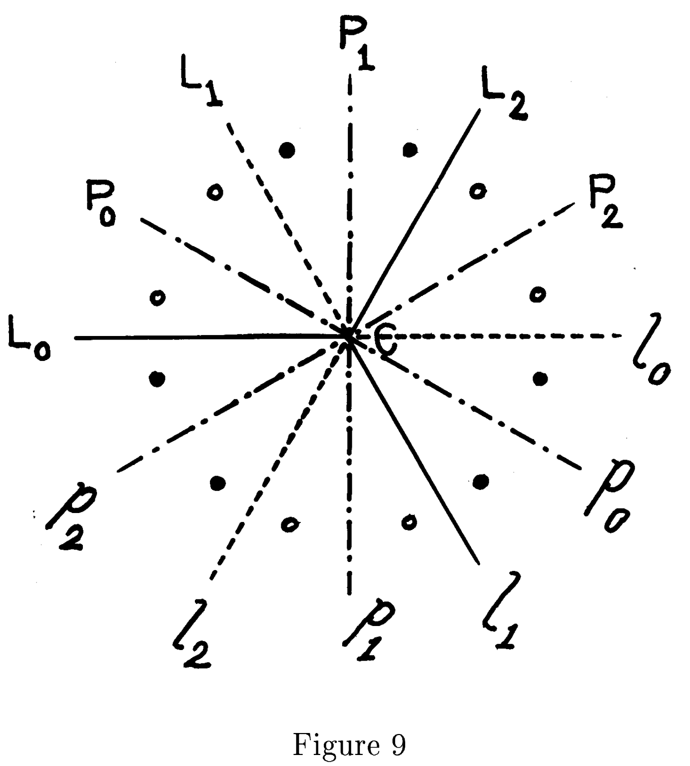

Figure 9

The binary axes CL_0, CL_1, CL_2, Fig. 9, then divide the half circumference of a circle, which is described around C as middle point, into $2q + 1$ equal parts. Here this number is an odd number, therefore one of the bisectors of the angle L_0CL_1, L_1CL_2 etc. will be normal to CL_0; consequently it will always give a symmetry plane normal to the axis CL_0. It is this the plane, whose trace displays on the plane of Fig. 9 the straight line P_1Cp_1, whereby the last plane is supposed normal to the major axis. The simultaneous existence of a symmetry plane and a binary axis, that stands normal to it, brings that of a symmetry centre with it (Statement XXII).

[44] Moreover the binary axes are all directly different with respect to the major axis, and consequently of the same kind. The white and black circles of Fig. 9 show the positions of the homologous vertices. The black circles mean the vertices placed under the drawing surface, the white ones those placed above this plane.

The symbols of the polyhedra, on which Statement XXXVII and XXXVIII relate, will therefore after the representation taken in S. 12 be:

$$\left[\Lambda^{2q+1}, (2q+1)L^2, C, (2q+1)P^2\right]$$

$$\text{and}\quad \left[\Lambda^{2q+1}, (q+1)L^2, 0C, \Pi, (2q+1)P\right].$$

One sees from Statement XXXIV, XXXV and XXXVIII, that the polyhedra with major axis of odd order can have only the seven kinds of symmetry, which are given in the table at the end of this treatise.

One can give q in this table all possible values, from $q = 1$ inclusively to $q = \infty$.

§ IV. Symmetric spheroidal polyhedra

Statement XXXIX. *Every spheroidal polyhedron has at least two axes L^q and $L^{q'}$, whose order numbers are greater than two.*

A spheroidal polyhedra (Definition IX) can not only have a sole symmetry axis, because otherwise, by avoiding the repetition of this axis through the symmetry planes of the polyhedron (Statement VIII, Addendum), these planes would have to contain the only symmetry axis or coincide with its normal plane. This sole axis could then be viewed as a major axis and the polyhedron would not be spheroidal.

There are as a consequent two or more symmetry axes in the spheroidal polyhedra.

Let L^q and $L^{q'}$ next be two axes, whose order numbers q and q' are not smaller than those of the other axes, therefore I maintain, that $q > 2$ and $q' > 2$.

We firstly assume, that one has $q = 2$ and $q' = 2$. The normal to the plane of the [45] axes L^2 and L'^2 will likewise be a symmetry axis $L^{q''}$ (Statement XI), and here $q'' > q$, $q'' > q'$ can not be the case, therefore $q'' = 2$. Moreover L^2 and L'^2 stand perpendicular to each other, otherwise one would have yet a third binary axis on the plane of the axes L^2 and L'^2, and q'' would be greater than 2 (Statement XIII), which is not possible. One has therefore three binary perpendicular axes L^2 ,L'^2, L''^2, and I maintain, that at least one of these axes can be looked at as a major axis.

In deed no other binary axis other than the three axes L^2, L'^2 and L''^2 can thus occur, because any axis diagonal to L^2 would, together with L^2, be another binary axes in the plane determined by itself and L^2 (Statement X, Addendum), and a symmetry axis of order higher than 2 would result (Statement XIII).

In exactly the same way any symmetry plane must pass through one of the three axes $L^2 \ldots$; without this another three homologous binary axes, with regard to this plane would arise, and the total number of the binary axes would become six, which we have already proved as impossible. Let P then be a symmetry plane, which passes through L^2. If a second symmetry plane exists, then it must be normal to P, otherwise their intersection line would be an axis, whose order would be higher than 2 (Statement XI). We will thus see, whether the position of these planes can bee such a one, that the polyhedron has no major axis.

If the plane P neither passes through L'^2, nor through L''^2, then P must go through L^2, otherwise its intersection straight-line with P would introduce a fourth binary axis, which is impossible. This is exactly the same with the other planes P'' and P''', which necessarily would all have to pass through L^2. Therefore in this case the axis L^2 would suffice the demands placed on the major axis, and the polyhedron would not be spherical.

If on the other hand the plane P did not contain only L^2, but yet one of the two binary axes L'^2 and L''^2 (say the axis L'^2), then the plane P', which must pass through the normal to the plane P, ie through L''^2, would contain the axis L^2 or the axis L'^2 (say the axis L^2), with this its intersection straight-line with P makes no fourth binary axis. If there is then yet a third plane P'', then the same must simultaneously be perpendicular to P and P' (compare above); it will therefore pass through L'^2 and L''^2, and no other symmetry plane can exist. In this case either one of the axes L^2, L'^2 and L''^2 [46] can be viewed as a major axis, and the polyhedron is not spheroidal.

Consequently one can not have $q = 2$ and $q' = 2$.

We now assume, that $q > 2$ and $q' = 2$. The two axes L^q, L'^2 will be perpendicular to each other, otherwise the axis L'^2 would force the axis L^q, to repeat itself at least once, and the number q' would be smaller than the order number of the two axes of the polyhedron, which from the assumption made earlier is not possible. On the same ground no further axis can be given outside the plane normal to L^q. Therefore the first condition, that L^q is a major axis, is therefore made sufficient.

Similarly the symmetry planes of the polyhedron, which are subjected to the condition, must not produce anew the axis L^q, necessarily contain L^q or be perpendicular to it. Therefore L^q will be a major axis, and the polyhedron no spheroidal.

It can not be the case, therefore, that $q > 2$ and $q' = 2$, consequently it must be that $q > 2$ and $q' > 2$.

Statement XL. *If in a polyhedron there exist two axes of order higher than the second one, then the polyhedron is necessarily spheroidal.*

Because if the polyhedron possessed a major axis, the other axes would necessarily be binary (Statement XV), which is against the assumption. Consequently the polyhedron is spheroidal.

Definition X. *The Statements XXXIX and XL show, that one can give yet another definition of the spheroidal polyhedra, than the one in S. 13, such that one says, these polyhedra being symmetrical polyhedra with more axes, at least two of which have a symmetry of higher order than the second one. The polyhedra with major axis could then be defined as polyhedra, which possess one or more symmetry axes, at most one of which is of higher [order] than the second order.*

Statement XLI. *In any spheroidal polyhedron with a [47] symmetry axis L^q of higher order than the second order, the total number Q of the axes of the order q, which belong to this polyhedron, must necessarily be equal to half of the value, which the number of vertices of a regular hull polyhedron can have, the condition following this suffices: 1. that the centre of its form at the same time be a symmetry centre; 2. that each of its space vertices of q sides be formed.*

The axis L^q is necessarily connected with an axis $L^{q'}$, whereby q' is greater than 2 (Statement XXXIX). If one now turns L^q around $L^{q'}$ by an angle $\frac{360°}{q'}$, one will have the location of a second axis of order q, which is different from the original axis L^q. (Statement X, Addendum).

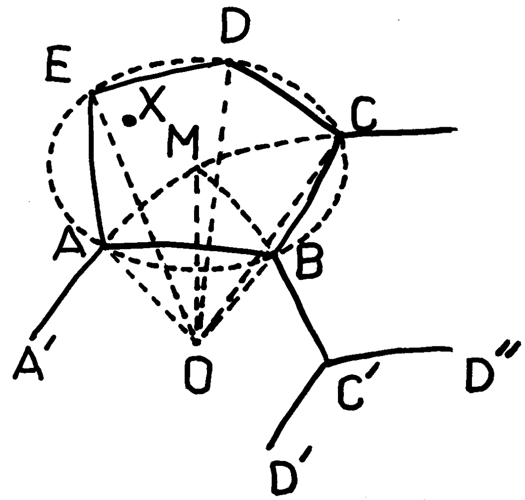

Figure 10

Let then OA and OB, Fig. 10, be these two axes of the order q, which meet in O, the middle point of the form of the polyhedron. From O as centre we describe the sphere of radius 1, which intersects the two axes OA and OB in A and B, and draw the arc AB of a greatest circle. One can always assume:

$$\operatorname{arc} AB < 90^\circ \quad \text{or} \quad = 90^\circ.$$

In the opposite case one would introduce the supplement of the angle AOB in the consideration. One can likewise always assume, that OA and OB be chosen such, that their inclination be the overall smallest of all of those, which the axes of order q have towards one another. This established, we turn the polyhedron by $\frac{360^\circ}{q}$ around the axis OB of order q. The point A will come to C: if we connect BC by the arc of the greatest circle, then

$$ABC = \frac{360^\circ}{q}$$

results, and the straight line OC will likewise be an axis of the order q. (Statement X, Addendum). If we draw in the same manner the arc of the greatest circle CD, so that

$$CD = CB = AB \qquad \text{and} \qquad BCD = \frac{360^\circ}{q},$$

then the straight line OD will also be an axis of the order q.

If one turns the polyhedron a second time by $\frac{360^\circ}{q}$ around the axis [48] OC and from B against D, then the effects of this second revolution is that, the point B goes on to D. The point A remains at C. The two revolutions are equivalent to a single revolution around the point M, the pole of the small sphere circle, which passes through the points A, B, C and D†. The second revolution around OB and OC does not alter the apparent position of the vertices of the polyhedron; as little does the single revolution around M, which replaces it, change this apparent position; therefore the straight line OM will be a symmetry axis of the polyhedron, and one sees, that, if one turns the polyhedron around OM by an angle, which amounts to the face angle AMC, the position of the vertices is not altered in the act. Consequently this angle is commensurate with the circumference (Statement II). Therefore the number of the vertices A, B, C, D etc, which is positioned on the circumference of the small circle $ABCD$, is a limited one: consequently these vertices makes a regular, registered polygon, whose side number can be represented by r. One will always be able to assume, that A and B are two neighbouring vertices; one can therefore write:

$$AMB = \frac{360^\circ}{r},$$

a formula, in which the number r necessarily must be greater than 2.

Here AMB and ABC are parts of 360°, therefore the regular spherical polygon $ABCDE$, which repeats itself in $CBC'D'' \ldots$ and in $A'ABC'D' \ldots$, covers in the end the whole surface of the sphere. The totality of the points thus contained will here make the vertices of a regular, registered polyhedron, and this regular polyhedron will necessarily be one of those, in which the sides meet together in a number q, to make each of their vertices.

The five regular polyhedra of geometry all, except the regular tetrahedron, have a symmetry centre in the centre of their form; but in that of the sphere registered tetrahedra the angle distance AB between two vertices exceeds the value 90°. This case therefore can not occur, as it is against our construction procedure.

[49] The registered polyhedron, which results from the construction above, will therefore either be the cube, ie correspond to the case $q = 3$, $r = 4$; one has then‡

$$AB = 70^\circ 32', \quad AM = 54^\circ 44';$$

† This pole M lies in the intersection point of the greatest circle arcs BM and CM, which bisect the spherical angle ABC and BCD.

‡ The arcs AB and AM are given by the well known formulae

$$\cos\frac{1}{2}AB = \operatorname{cosec}\frac{\pi}{q}\cos\frac{\pi}{r} \quad \text{and} \quad \cos AM = \cot\frac{\pi}{q}\cot\frac{\pi}{r}.$$

or the regular octahedron, what corresponds to the case $q = 4$, $r = 3$; one has then:

$$AB = 90^\circ, \quad AM = 54^\circ 44';$$

or the regular dodecahedron, which corresponds to the case $q = 3$, $r = 5$; then

$$AB = 41^\circ 49', \quad AM = 37^\circ 23';$$

or the regular icosahedron, which corresponds to the case $q = 5$, $r = 3$; then

$$AB = 63^\circ 26', AM = 37^\circ 23'.$$

Let M now be the number of the vertices of the regular polyhedron thus contained; here each vertex has a homolog, which lies diametrically opposite to it, therefore one sees, that the total number Q of the axes of the order q will be at least equal to $\frac{1}{2}M$. I maintain besides, that $Q > \frac{1}{2}M$ can not be. Because should $Q > \frac{1}{2}M$, then one of the axes of the order q meet the sphere in a point X, lying in the inside of one of the spherical polygons $ABCDE$. One of the angular distances between X and the vertices A, B, C, D would necessarily be smaller than AM and all the more so AB (from the comparative table of the related values of AB and AM), which the assumption of the least inclination of the two axes OA and OB reiterates. It can not be therefore that

$$Q > \frac{1}{2}M,$$

consequently

$$Q = \frac{1}{2}M.$$

[50] **Statement XLII.** *A spheroidal polyhedron can only have ternary, quaternary or quinary axes, the binary axes not included.*

This follows from the content of the previous Statement. Here L^q is one of the axes of the polyhedron, therefore the number q can only be the number of sides, which can connect one another, to make the vertex of a regular polyhedron; consequently one must have:

$$q = 3 \quad \text{or} \quad 4 \quad \text{or} \quad 5.$$

Statement XLIII. *There are two different groups of spheroidal polyhedron, that, which contain four ternary axes, and that, which contain ten ternary axes.*

We wish by mutually examining the four cases, to whose opposite generation the Q axes L^q goes, and let M always be the number of the vertices of the regular, registered polyhedron, on to which this kind of repetition goes.

In the case of the cube (Statement XLI)

$$q = 3, \quad M = 8, \quad Q = \frac{1}{2} = 4.$$

In the case of the octahedron

$$q = 4, \quad q = 4, \quad M = 6, \quad Q = \frac{1}{2}M = 3.$$

The three quaternary axes, which one thus obtains, are perpendicular to one another, there are thus four ternary axes (Statement XIV), and it can gives no greater number, because the new ternary axes would force the quaternary axes, to repeat themselves, so that $Q > 3$ would result, which is impossible.

In the case of the dodecahedron

$$q = 3, \quad M = 20, \quad Q = \frac{1}{2}M = 6.$$

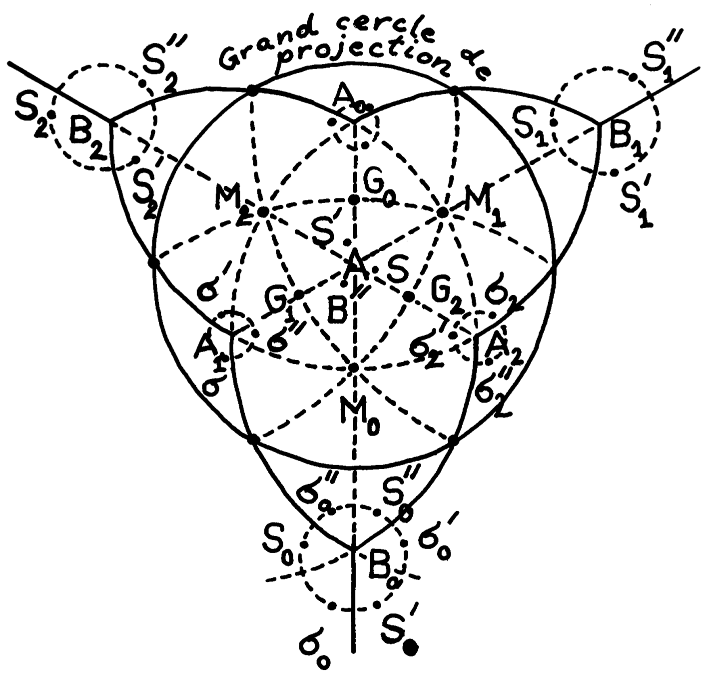

Figure 11

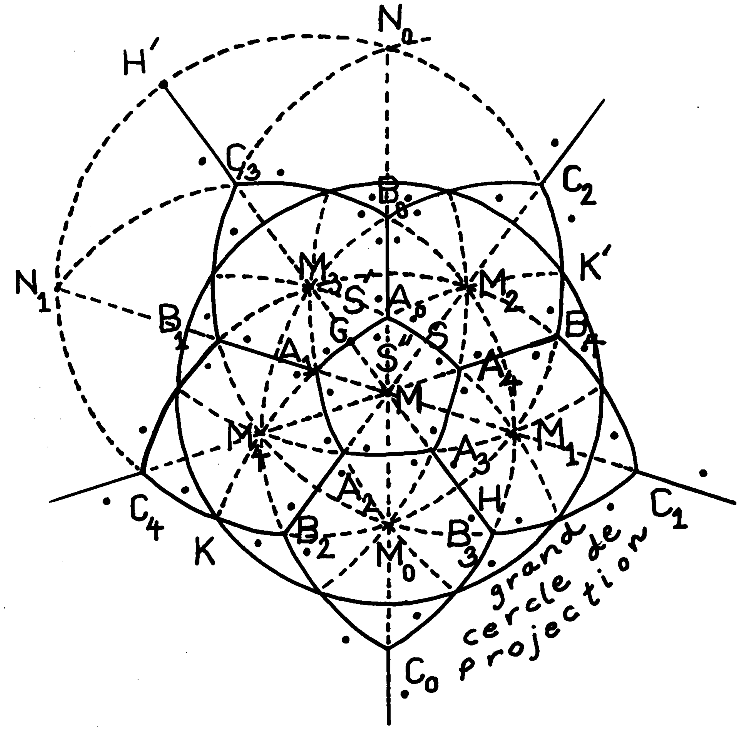

Figure 12

[51] Let next M, M_0 and M_1, Fig. 12, be three neighbouring vertices of a registered icosahedron. The normal dropped from the centre of the sphere on the side MM_0M_1 is apparently a ternary axis, and here the icosahedron has twenty pairwise parallel faces, therefore it will have ten ternary axes; but it can have no greater number of these, because $q = 3$ one only finds the two values $Q = 4$ and $Q = 10$ (Statement XLI) consequently etc.

Addendum. We can therefore divide the spheroidal polyhedra into two groups: the *quaterternary* ones with four ternary axes, which are ordered as the four major diagonals of a cube, and the *decemternary* ones with ten ternary axes, which are ordered as the ten major diagonals of a regular dodecahedron.

Quaterternary polyhedra

Statement XLIV. *If one builds a cube, which has as diagonals the four ternary axes of a given quaterternary polyhedron, then the three normals* *dropped from the centre of the form on the side faces of this cube are, for the polyhedron three axes of the same kind, and have binary or quaternary symmetry.*

One describes around the centre the form of the polyhedron, the intersect point of its four ternary axes, as midpoint, with the unit as radius, a sphere, which the upper ends of our four ternary axes will intersect in A, A_0, A_1 and A_2, Fig. 11. (I have projected the surface of this sphere stereographically on the plane of the greatest circle, whose pole is A; the reader is however requested, therefore to follow the picture, as if he moves on the sphere. The centre point O of which is not shown on the figure.) The registered cube, whose four upper vertices are A, A_0, A_1 and A_2, will have in B_0, B_1 and B_2 the vertices lying diametrically opposite to A_0, A_1 and A_2. $AA_0B_1B_2$, $AA_0B_2A_1$ and $AA_1B_0A_2$ are three spherical quadratic polygons, whose midpoint M_0, M_1 and M_2 are the endpoints of the three normals, which are dropped from the midpoint on the side faces of the registered cube.

One double turn of 120°, namely first around A_2 as pole, from A on to B_0, and secondly around B_0 as pole, from A_2 [52] on to A_1, A goes on B_0 and A_2 goes on A_1, without changing the position of the vertices. This double turn is equivalent to one turn of 180° around the pole M_0. Therefore OM_0 is an axis, whose symmetry is 2 or a multiple of 2. Besides, the symmetry of OM_0 can not be of higher order than 4; otherwise the number of the ternary axes, which are placed around the pole M_0, would exceed 4, which contradicts the starting conditions. Therefore the three perpendicular axes OM_0, OM_1 and OM_2 are binary or quaternary.

Statement XLV. *The quaternary polyhedra, with binary axes perpendicular to one another, can have no other binary axes.*

If another binary axis existed, then it would be able to cut through the upper part of the sphere surface, Fig. 11, only in one of the three points, which the midpoint, are either the curves AA_0, AA_1 and AA_2, or the homologous curves A_0B_1, A_0B_2, A_1B_2, A_1B_0, A_2B_0 and A_2B_1 then for every other position this axis would force it to repeat the ternary axes, which would double its number. If we assume, that G_0, the centre of AA_0, be the endpoint of the new binary axis, then two turns of the polyhedron, firstly of 180° around the pole G_0, secondly of 120° around the pole A_0 in the sense A towards B_1, bring

A to A_0, then to A_0,
A_0 to A, then to B_1,
B_1 to A_1, then to A_2,
A_2 to B_2, then to A.

These two turns, which leave the place of the vertices of the polyhedron unchanged, are equivalent to a single turn of 90° around M_1 in the sense A

towards A_0. Then the pole M_1 would thus be the end of a quaternary axis, which contradicts the initial conditions.

Therefore no other binary axis can exist.

Remark. Let S, Fig. 11, be a vertex of the given polyhedron. One can assume, that this axis is found on the surface of the sphere, in which one takes its distance OS to the centre of the form of the polyhedron as unit; its two homologs with reference to the ternary pole A are S' and S''. The system $SS'S''$ will repeat [53] itself in succession of the binary nature of the pole M_0, M_1 and M_2 in $S_0S'_0S''_0$, $S_1S'_1S''_1$ and $S_2S'_2S''_2$. The complete system of the homologs of any one vertex will therefore in the present case be a polyhedron, which one can register to the sphere.

Addendum. From the arrangement of the twelve vertices of this polyhedron it easily turns out, that it has neither symmetry planes nor a symmetry centre, at least in the general case, where the vertex offers no special feature with regard to its position in the inside of the spherical triangle A_0A_2A. (Compare the proofs of the two following statements.)

Statement XLVI. *The quaterternary polyhedra with binary, right-angled axes can either have six symmetry planes, which coincide with the six planes pairwise connecting the ternary axes, or otherwise three symmetry, which coincide with the three planes pairwise connecting the binary axes. Any other symmetry planes can not likewise be so.*

Every other location of a symmetry plane, from those just mentioned, would cause the four ternary axes to repeat themselves, and must consequently be dismissed.

If A_1AM_1, Fig. 11, depicts one of the symmetry planes, then the nature of the pole A demands that A_1AM_2 and A_0AM_0 also be likewise; the ternary nature of the pole A_1 forces us to extend this conclusion to the planes $B_2A_1M_0$ and $B_0A_1M_2$, and the same applies to $A_0M_1A_2$ in view of the binary axis OM (Statement XXIII.)

In this case every one of the triangles $SS'S''$, $S_0S'_0S''_0$, $S_1S'_1S''_1$ and $S_2S'_2S''_2$ is replaced by a hexagon. We have further confined ourselves to portray the one which encloses the vertex B_0. The twenty-four vertices of the polyhedron can be reduced to twelve, if the vertex S, which is viewed as important for the positions of all others, fell on one of the three greatest arcs A_0AM_0, A_1AM_1, A_2AM_2. These vertices can be reduced to four, if S coincided with A etc.

Let us assume, that $M_2G_0M_1$ represents a symmetry plane; [54] then, as a consequence of the ternary nature of the pole A, would also $M_1G_2M_0$ and $M_0G_1M_2$ be symmetry planes. In that case the triangle $SS'S''$ repeats itself around the pole A_0, A_1, and A_2. We have confined ourselves thereupon, to portray this kind of repetition around the pole A_1. The triangle $\sigma\sigma'\sigma''$ is then the homolog of the triangle $SS'S''$ with respect to the symmetry plane $M_2G_1M_0$.

Incidentally these two systems of symmetry planes can not occur at the same time, because it would otherwise give four symmetry planes, which intersect one another at the pole M_0, and the axis OM_0 would then be at least quaternary (Statement XI), which contradicts conditions made in the statement.

Statement XLVII. *The quaternary polyhedra with binary right-angled axes can only have one symmetry centre under the condition, that they have three symmetry planes, which pairwise join the binary axes, and vice versa the existence of these planes entails the same of the symmetry centre.*

This is a consequence from Statement XXI and XXII.

Statement XLVIII. *The quaternary polyhedra with binary right-angled axes can only have three different kinds of symmetry, depending on whether they have no symmetry planes, or six going through the ternary axes, or three symmetry planes going through the binary axes.*

This is a consequence from the addendum to Statement XLV and from Statement XLVI. In view of Statement XLVII and the definition, which we made S 12, one has the three symbols:

$$\begin{gathered}[4L^3, 3L^2, 0C, 0P],\\ [4L^3, 3L^2, C, 3P^2],\\ [4L^3, 3L^2, 0C, 6P].\end{gathered}$$

Statement XLIX. *Every quaterternary polyhedron with quaternary axes has six binary axes, which the opposite edges of a cube, of which the four ternary axes of the polyhedron are diagonals, pairwise join.*

[55] I maintain, that, if one connect the midpoint O of the sphere, Fig. 11, with the point G_0, the middle of AA_0, this straight line must be a binary axis of the polyhedron.

If we give the polyhedron one turn of 90° around OM_1 from A into A_0, and a second turn of 120° around OA_0 from B_1 into A, then these double movements, which do not change the apparent position of the vertices, go from

$$\begin{aligned}
&A \ \text{ to } A_0, \text{ then to } A_0,\\
&A_0 \text{ to } B_1, \text{ then to } A,\\
&B_1 \text{ to } A_2, \text{ then to } A_1.\\
&A_2 \text{ to } A, \ \text{ then to } B_2.
\end{aligned}$$

The result of these two movements is the same, as if one had turned the polyhedron 180° around G_0; thus OG_0 is an axis of even order, that obviously can only be binary, and the same would be the case for the five remaining straight lines, the homologs of OG_0. Three of the six binary axes are situated in the plane of the base circle of the projection of the sphere.

One could prove, as has already been done in the proof of Statement XLV, that every other straight line would be unsuitable, to be a binary axis of the system.

Statement L. *The quaterternary polyhedra with quaternary axes have, if they in general have symmetry planes, of which necessarily six, which pass through the ternary axes, and three, which pass through the quaternary axes; they have at the same time a symmetry centre.*

I will indicate the planes, which pass through the quaternary axes with P^4 and those, which pass through the ternary axes, with P^2. One has already earlier seen, (with the proof of Statement XLVI), that these are the only possible symmetry planes.

If we accept the existence of the planes P^4; $M_0G_1M_2$ and $M_0G_2M_1$, Fig. 11, will be two symmetry planes, which meet each other in a quaternary axis, consequently AM_0B_0 and $A_1B_0A_2$ will likewise be symmetry axes (Statement XXV). The system P^2 therefore joins the system P^4. One would prove in exactly the same way, that the system of planes P^2 always presupposes the coexistence of the system of planes P^4.

Each of the three planes of the system P^4 is normal to one of the three quaternary axes. [56] The six planes of the system P^2 are each normal to one of the six binary axes; because if in a cube with four vertical edges, one joins the centres of two mutually opposing ones of these four edges, then each of these straight lines will be normal to the plane, which passes through the two other edges. The existence of the symmetry planes furthermore entails that of the symmetry centre (Statement XXII).

Statement LI. *With the quaterternary polyhedra with quaternary axes are only admissible two kinds of symmetry, depending on whether they have symmetry planes or not.*

This follows from the previous Statement.

In the case, where the polyhedron has no symmetry plane, it can have no symmetry centre, as consequence of Statement XXI. The complete system of the homologs of the vertex S, Fig. 11, builds then a registered polyhedron of twenty-four vertices, which are grouped into three's around each one of the eight poles A, A_0, A_1, A_2, B_0 etc. One has with the Figure confined oneself to showing the triangle $\sigma_2\sigma_2'\sigma_2''$, which surrounds the pole A_2 in this case.

If the polyhedron has its nine symmetry planes, then the eight triangles $SS'S''$, $S_0S_0'S_0''$ etc. are replaced by eight hexagons, and the system of the homologs contains forty-eight vertices.

The symbol of these two kinds of symmetry are the following:

$$\begin{gathered}[3L^4, 4L^3, 6L^2, 0C, 0P] \\ [3L^4, 4L^3, 6L^2, C, 3P^4, 6P^2].\end{gathered}$$

Decemternary polyhedra

Statement LII. *If one constructs a regular dodecahedron which has as diagonals the ten ternary axes of a given decemternary polyhedron, then the six normals which fall from the centre of the form on the side faces of this dodecahedron are quinary symmetry axes for this polyhedron.*

With the unit as radius, and the common intersection of the ten ternary axes as midpoint, one describes a sphere which in A_0, A_1, [57] A_2, A_3, A_4, B_0, B_1, B_2, B_3, B_4, Fig. 12, intersects the upper half of the ten ternary axes. (I have stereographically projected the surface of this sphere on the plane of the greatest circle which lies parallel to the face $A_0A_1A_2A_3A_4$ of the registered regular dodecahedron whose vertices are these end points A_0, A_1, …, B_0[,] B_1 etc. The reader is asked to imagine the points and lines of the figure on the surface of the sphere; the midpoint O of the sphere is not given on the drawing.) By pairwise connecting the intersection point, regular spherical pentagons result whose twelve midpoints M, M_0, M_1, …, N_0, N_1 etc. are the extreme ends of the same radii which stretches from the midpoint of the sphere normal to the twelve side faces of the dodecahedron.

Two turns of 120°, one around OA_0, from A_4 towards B_0, the other around OB_0 from A_0 towards C_2, A_4 will go on to B_0 and A_0 on to C_2. These two turns, which do not differ from the position of the vertices of the polyhedron, are of the same value with an only turn of 144° around OM_2 from A_0 towards B_0; this would, if repeated three times, be the same as one turn 72°; therefore the axis OM_2 is a quinary axis, and the same holds with OM, OM_0, OM_1, etc. The points M_0, M_1, etc. are the vertices of a regular icosahedron described by the sphere.

Statement LIII. *The decemternary polyhedra always have fifteen binary axes.*

One turns the given polyhedron 72° around OM, Fig. 12, from A_0 onto A_1, then 120° around OA, from A_2 onto A_0; as consequence of these two turns the pole A_0 comes over A_1 and A_1 comes over A_0. The end result is the same, as if the polyhedron were turned 180° around the radius OG, which ends in the middle of the curve A_0A_1. Now the apparent places of the vertices are not changed; consequently G is the end point of an axis of even order, that is obviously a binary axis. Here the regular dodecahedron has thirty pairwise opposite sides, therefore the total number of binary axes is equal to fifteen.

No other diameter of the sphere can be a binary axis, because which ever one likes to be in its position, therefore it would force itself to repeat the ternary axes, and one would obtain more than ten ternary axes, that which we know is impossible. (Statement XLIII).

[58] One could also obtain the fifteen binary axes, in which one pairwise joins the centres of the edges facing one another of the icosahedron $MM_0M_1\ldots$

Statement LIV. *The decemternarial polyhedra can have as symmetry planes the fifteen planes, which go through the six quinarial axes pairwise joined. In the opposing cases these polyhedra have no symmetry plane at all.*

Let us take a look specially at a plane, which goes through the centre of the sphere and through the vertices M and M_3, Fig. 12.

This plane becomes a symmetry plane for the point system (A_0, A_1), (A_4, A_2), …, (M_2, M_4), M_1, M_0 etc.; it therefore does not result in the doubling of the number of axes, consequently it answers nothing about the existence of such a symmetry plane.

Now lies the plane MM_3 normal to the straight line KK', which is a binary axis of the system. The homologs of this plane are in total fifteen in number, namely MM_0, MM_1, MM_2, MM_3, MM_4; M_0M_2, M_1M_3, M_2M_4, M_3M_0, M_4M_1; M_0M_1, M_1M_2, M_2M_3, M_3M_4, M_4M_0. These are obviously the only symmetry planes, which the polyhedron can have; for any other position the number of the ternary axes would exceed 10, which can not be the case. (Statement XLIII)

Figure 12 shows the arrangement of the sixty homologous vertices of S around the pole A_0, A_1, etc. in the case where the polyhedron has no symmetry plane at all.

If the fifteen symmetry planes, which are mentioned above, exist in the polyhedron, then the triangle $SS'S''$ is replaced by a hexagon. For the figure not to be overcrowded, one has restricted oneself to showing only one of the hexagons, the one which would enclose the pole B_0. The system of the homologs of S then contains hundred and twenty vertices. For certain special positions of S this number can be reduced to sixty, to twenty, or to twelve vertices. This last case occurs when the vertice S is placed at the end of one of the quinary axes of the system.

Statement LV. *When the decemternary polyhedron has the fifteen symmetry planes, which are given in the previous statement, then these planes are [59] normal to the fifteen binary axes, and the polyhedron has a symmetry centre. In the contrariwise case it has none.*

It follows from the proof of the statement above, that the binary axis KOK', Fig. 12, is normal to the plane M_3GMA_3. Now this plane is actually one of the fifteen symmetry planes of the polyhedron, consequently each one of these planes is normal to one of the fifteen binary axes, therefore the polyhedron then has a symmetry centre (Statement XXII). If the polyhedron were without symmetry planes, then it would follows that its binary axes have no symmetry centre. (XXI)

Statement LVI. *The decemternary polyhedra have two different kinds of symmetry, depending on whether it has a symmetry centre or not.*

This is a consequence of Statements LIV and LV.

The symbols of these two types are:

$$\begin{gathered}\left[6L^5, 10L^3, 15L^2, 0C, 0P\right],\\ \left[6L^5, 10L^3, 15L^2, C, 15P^2\right].\end{gathered}$$

Statement LVII. *The axes, which characterise the symmetry of the quaternary polyhedra with binary right-angled axes, also enter into the symmetry of the decemternary polyhedra.*

Let us choose an arbitrary point of a ternary axis, eg. A_2, Fig. 12. The centre G of one of the two sides A_1A_0, A_3A_4, the side A_0A_4 lying opposite to the vertex A_2 in the pentagon $A_0A_1A_2A_3A_4$ of which closely fit, will be the endpoint of a binary axis (Statement LIII). The same will be the case with the points H and K, which are the homologs of G with regard to the ternary axis OA_2. In the spherical triangle HGK the three angles H, G and K are right [angles]. Therefore the three sides HG, KG and HK are equal to 90°.

The three axes OG, OH and OK are consequently three right-angled binary axes, and the spherical triangle GHK is such a one with three right angles. The vertex A_2 is the midpoint of this triangle. In exactly the same way A_4 will be the mispoint of the triangle GHK' likewise possessing three right angles, B_0 that [60] of the triangle $GK'H'$ obtained in exactly the same way, when H' is the lower end of the axis OH, and B_1 the midpoint of the triangle KGH' with the three right angles.

The four ternary axes OA_2, OA_4, OB_0, OB_1 then combine to one another with the three binary right-ngled axes in the same position, which characterises the quaterternary polyhedra with binary right-angled axes.

Remark. One could replace the combination OA_2, OA_4, OB_0, OB_1 with one of the following four combinations.

$$[OA_0, OA_3, OB_1, OB_2], \qquad [OA_0, OA_2, OB_3, OB_4]$$
$$[OA_1, OA_3, OB_0, OB_4], \qquad [OA_1, OA_4, OB_2, OB_3].$$

Statement LVIII. *The polyhedra* $\left[6L^5, 10L^3, 15L^2, 0C, 0P\right]$ *have all elements of the symmetry of the polyhedra* $\left[4L^3, 3L^2, 0C, 0P\right]$. *The polyhedra* $\left[6L^5, 10L^3, 15L^2, C, 15P^2\right]$ *have all elements of the symmetry of the polyhedra* $\left[4L^3, 3L^2, C, 3P^2\right]$.

The part relating to the symmetry axes of this statement has already been proved in the previous Statement. One concludes from this easily, that the polyhedra $\left[6L^5, 10L^3, 15L^2, 0C, 0P\right]$ have all elements of the symmetry of the polyhedra $\left[4L^3, 3L^2, 0C, 0P\right]$.

If the decemternary polyhedron [has] fifteen symmetry planes besides, then the face KG, GH, HK of Fig. 12 will be below it, and the planes $3P^2$ will represent the polyhedron $\left[4L^3, 3L^2, C, \;\; 3P^2\right]$. Here the centre of the symmetry C in both cases exists, therefore one sees, that the symmetry characterised by $\left[4L^3, 3L^2, C, 3P^2\right]$ is included with the polyhedra $\left[6L^5, 10L^3, 15L^2, C, 15P^2\right]$ complete.

Remark. The theorem LVII and LVIII are for the general theory of the symmetrical polyhedra only indirectly imortant. They are here taking into consideration the application given, which one can make of in the crystallography with the study of regular system.

[61]
Classification of the polyhedra after the type of their symmetry with information of the minimum number of their vertices

Polyhedron			*Symbol of the symmetry of the polyhedron*	*Class of the poly-hedron*	*Minimum number of vertices of the* [polyhedron] 1st *type*	2nd *type*	3rd *type*	4th *type*
	asymmetrical		$0L, 0C, 0P$	1st	1	1	1	1
		without	$0L, C, 0P$	2nd	2	2	2	
		axes	$0L, 0C, P$	3rd	1	1	1	
			$\Lambda^{2q}, 0L^2, 0C, 0P$	4th	$2q$	$2q$		
		of	$\Lambda^{2q}, 0L^2, C, \Pi$	5th	$2q$	$2q$		
		even	$\Lambda^{2q}, qL^2, qL'^2, 0C, 0P$	6th	$4q$			
		order	$\Lambda^{2q}, 0L^2, 0C, qP, qP'$	7th	$2q$	1		
	with		$\Lambda^{2q}, qL^2, qL'^2, C, \Pi, qP^2, qP'^2$	8th	$2q$	0 or $2q$*		
	one		$\Lambda^{2q}, 2qL^2, 0C, 2qP$	9th	$4q$			
	major		$\Lambda^{2q+1}, 0L^2, 0C, 0P$	10th	$2q+1$	$2q+1$		
	axis		$\Lambda^{2q+1}, 0L^2, C, 0P$	11th	$4q+2$	$4q+2$		
symmetrical		of	$\Lambda^{2q+1}, 0L^2, 0C, \Pi$	12th	$2q+1$	$2q+1$		
		odd	$\Lambda^{2q+1}, (2q+1)L^2, 0C, 0P$	13th	$4q+2$			
		order	$\Lambda^{2q+1}, 0L^2, 0C, (2q+1)P$	14th	$2q+1$	1		
			$\Lambda^{2q+1}, (2q+1)L^2, C, (2q+1)P^2$	15th	$4q+2$			
			$\Lambda^{2q+1}, (2q+1)L^2, 0C, \Pi, (2q+1)P$	16th	$2q+1$			
			$4L^3, 3L^2, 0C, 0P$	17th	12			
			$4L^3, 3L^2, C, 3P^2$	18th	12			
	quarterternary		$4L^3, 3L^2, 0C, 6P$	19th	4			
spheroidal			$3L^4, 4L^3, 6L^2, 0C, 0P$	20th	24			
			$3L^4, 4L^3, 6L^2, C, 3P^4, 6P^2$	21st	6			
	decemternary		$6L^5, 10L^3, 15L^2, 0C, 0P$	22nd	60			
			$6L^5, 10L^3, 15L^2, C, 15P^2$	23rd	12			

Above table gives a classification of the polyhedron in twenty three classes following the principles set forth in this treatise. For the understanding the symbols Λ, L, L', C, Π, P, P' compare p.12 [definition VIII].

One will notice, that the classes 4, 5 inclusively to 16 again are divided into orders of different types, depending on the order number of the symmetry of the major axis.

[62]

One wish for example to know from this table the integral elements of the symmetry of a polyhedron of the 7th class 4th order. Its symbol becomes:

$$\left[\Lambda^4, 0L^2, 0C, 2P, 2P'\right]$$

from which one sees that this polyhedron has one quaternary axis, four symmetry planes going through this axis, which intersect one another less than 45°, two planes of a first kind, which are right-angled to one another, and two other planes also right-angled to one another, but of a different type from the last ones—in comparison no binary axis and no symmetry centre.

The four last columns show the smallest number of vertices of each polyhedron. All vertices of the same type make up a system of homologous vertices and there are as many such homologous systems as [there are] different types of vertices in the polyhedron.

Through a little reflection one will easily find the form, which the polyhedron with the smallest number of vertices must have. Therefore the simplest polyhedron becomes:

in the 1st class the non regular tetrahedron;

in the 2nd class the non regular octahedron with parallelogrammatical bases;

in the 3rd class the scalene triangle;

in the 19th class the regular tetrahedron;

in the 21st class the regular octahedron;

in the 23rd class the regular icosahedron *etc*;

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Kinder is a German word that means 'children'. So the name of the place should more correctly be written 'Kinderscout'. But then again there are Kinder Downfall and Kinder Low. To do the same thing everywhere would probably result in something seemingly out of place sitting in an English context. Therefore the name is normally written 'Kinder Scout'.

Kit Tyabandha
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